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INVERSE PROBLEMS OF DEFORMATION OF NONLINEAR VISCOELASTIC BODIES

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We consider two classes of inverse problems of finding, in a given time interval $[0, t_*]$, external force and kinematic actions which ensure, for $t = t_*$, the required residual displacements of the surface points of a physically nonlinear viscoelastic body under zero external loads at this moment. The correctness of the corresponding formulations is shown, iteration methods of solution are substantiated, and the convergence rate of successive approximations to an exact solution is estimated. In addition, the level of residual stresses in a body at moment $t = t_*$ and the change in the level of residual displacements of the body for $t > t_*$ in the absence of external forces owing to residual-stress relaxation are estimated.

1. Formulation of Inverse Problems. Let a uniformly heated body occupy the region V with the boundary S subject to the necessary smoothness condition [1]. Equations which determine the process of its deformation are of the form

$$\varepsilon_{kl} = a_{klmn}\sigma_{mn} + \varepsilon^c_{kl}, \tag{1.1}$$

where ε_{kl}, ε^c_{kl}, and σ_{kl} are the components of the full- and viscous-strain and stress tensors, respectively; a_{klmn} are the elastic-compliance tensor components possessing the known properties of symmetry and positive definiteness [1], i.e.,

$$a_{klmn}\sigma_{kl}\sigma_{mn} \geqslant a\sigma_{kl}\sigma_{kl}, \qquad a > 0. \tag{1.2}$$

Here and below, the usual rule of summation over repeat subscripts is adopted; k, $l = 1$, 2, and 3. The strains ε_{kl} are small and are expressed via the components of the displacement vector u_k by the following Cauchy relations:

$$\varepsilon_{kl} = (1/2)(u_{k,l} + u_{l,k}) \tag{1.3}$$

(the subscript after the comma refers to the derivative with respect to the corresponding coordinate).

The components of the viscous-strain rates $\dot\varepsilon^c_{kl} = \eta_{kl}$ are continuous stress functions:

$$\eta_{kl} = \eta_{kl}(\sigma_{mn}). \tag{1.4}$$

Relations (1.1) and (1.4) describe fairly satisfactorily the isothermal process of creep of nonstrengthening materials which have no first stage of creep. Among them, there are, for example, metallic materials at low levels of stress (i.e., in fairly prolonged deformation processes) and in short-time creep under conditions of high temperature and high levels of stress [2].

We assume that the functions (1.4) are subject to the following condition which generalizes the stability postulate [3]:

$$\Delta\eta_{kl}\Delta\sigma_{kl} \geqslant \lambda a_{klmn}\Delta\sigma_{kl}\Delta\sigma_{mn}, \qquad \lambda = \text{const}, \qquad \lambda > 0$$
$$[\Delta\sigma_{kl} = \sigma^{(1)}_{kl} - \sigma^{(2)}_{kl}, \qquad \Delta\eta_{kl} = \eta_{kl}(\sigma^{(1)}_{mn}) - \eta_{kl}(\sigma^{(2)}_{mn})]. \tag{1.5}$$

One can easily see that, by virtue of (1.2) and (1.5), relations (1.4) are reversible, i.e., the stresses $\sigma_{kl} = \sigma_{kl}(\eta_{mn})$ are uniquely determined by the known velocity components η_{kl}. [Indeed, for $\Delta\eta_{kl} = 0$, from

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(1.2) and (1.5) it follows that $\Delta\sigma_{kl}\Delta\sigma_{kl} \leqslant 0$. This is possible only for $\Delta\sigma_{kl} = 0$.] These functions are assumed to be subject to the following condition, which is similar to (1.5):

$$\Delta\eta_{kl}\Delta\sigma_{kl} \geqslant \lambda_1 b_{klmn}\Delta\eta_{kl}\Delta\eta_{mn}, \qquad \lambda_1 = \text{const}, \qquad \lambda_1 > 0. \tag{1.6}$$

Here b_{klmn} are the components of the elastic-modulus tensor which is inverse to a_{klmn}, so that

$$a_{klmn}b_{klij} = \delta_{im}\delta_{jn}, \tag{1.7}$$

where δ_{kl} are the unit-tensor components.

Note that dependences (1.4) can be generalized by introducing the time t into the right-hand side: $\eta_{kl} = \eta_{kl}(\sigma_{mn}, t)$. This corresponds to the so-called aging theories [2], which, despite the known shortcomings, describe well the creep process under constant or slowly varying stresses. In this case, in formulas (1.5) and (1.6), there should be $\lambda = \lambda(t) > 0$ and $\lambda_1 = \lambda_1(t) > 0$, respectively. Moreover, in the interval $[0, t_*]$, we shall assume hereinafter that these functions are constant, and, to this end, it suffices to set $\lambda = \min\limits_{0 \leqslant t \leqslant t_*} \lambda(t)$ and $\lambda_1 = \min\limits_{0 \leqslant t \leqslant t_*} \lambda_1(t)$.

For the most general dependences of the form of (1.4), the necessary and sufficient conditions for satisfaction of inequalities (1.5) and (1.6) in isotropic media can be obtained using the results of [3, Chapter 3] and taking into account the fact that (1.5) and (1.6) are equivalent, respectively, to the conditions that the quadratic forms are not negative:

$$\left(\frac{\partial\eta_{kl}}{\partial\sigma_{mn}} - \lambda a_{klmn}\right)\xi_{kl}\xi_{mn}, \qquad \left(\frac{\partial\sigma_{kl}}{\partial\eta_{mn}} - \lambda_1 b_{klmn}\right)\xi_{kl}\xi_{mn}.$$

As an illustration, for anisotropic media, we give the following potential dependences: $\eta_{kl} = \partial\Phi/\partial\sigma_{kl} = \eta\partial s/\partial\sigma_{kl}$, $\Phi = \Phi(s)$, $\eta(s) = \Phi'(s)$, and $s^2 = a_{klmn}\sigma_{kl}\sigma_{mn}$ for which (1.5) and (1.6) hold if [4] $\lambda \leqslant (\eta/s, \eta') \leqslant \lambda_1^{-1}$.

The known functions $\eta = B[\exp(\lambda s/B) - 1]$, $\eta = B\sinh(\lambda s/B)$, and $\eta = Bs/(B/\lambda - s)$ satisfy, for example, the latter inequalities ($B = \text{const}$) if one sets $\lambda_1^{-1} = \eta'(s_*)$, where s_* is the largest possible value of s, for example, $s_* = \sigma_y$ (σ_y is the yield point, because plastic strains are absent), i.e., $s < \sigma_y$.

The inverse problem of deforming a body in a natural state for $t < 0$ into a given residual state can be formulated as follows: which external actions should be exerted on the surface S (in the absence of mass forces) within the interval $[0, t_*]$ in order that at the moment $t = t_*$ the external forces are absent and the residual displacements $\tilde{u}_k$ of the surface points S take on the given values of $\tilde{u}_{k*}$?

We shall separate concrete classes of external force and kinematic actions when the loads or the displacements over the surface S vary, in the interval $[0, t_0]$, by a given-in-time law but with the unknown values for $t = t_0$. Within the interval $[t_0, t_*]$, the external forces are then lifted, i.e., a slow unloading occurs, so that, for $t = t_*$, the residual displacements are $\tilde{u}_k = \tilde{u}_{k*}$ on the surface S. Hence, we shall consider the following problems.

Problem 1. *Find the quantities p_{k0} such that under external loads $p_k = f(t)p_{k0}$ applied to the surface S, where $f(t)$ is a given nonnegative function $[f(0) = 0,\ \dot{f}(0) > 0,\ \dot{f}(t) \geqslant 0\ (0 < t \leqslant t_0),\ f(t_0) = 1,\ \dot{f}(t) \leqslant 0$ $(t_0 < t < t_*)$, and $f(t_*) = 0]$, the following condition is satisfied: for $t = t_*$, the residual displacements are $\tilde{u}_k = \tilde{u}_{k*}$ on S.*

Problem 2. *Find the quantities u_{k0} such that upon displacements $u_k = \varphi(t)u_{k0}$ $(0 \leqslant t \leqslant t_0)$ on S and under subsequent unloading when the external loads are $p_k = f(t)p_{k0}$ $(t_0 \leqslant t \leqslant t_*)$, the following condition is satisfied: the residual stresses $\tilde{u}_k = \tilde{u}_{k*}$ on S for $t = t_*$. Here $\varphi(t)$ and $f(t)$ are given functions $[\varphi(0) = 0$, $\varphi(t_0) = f(t_0) = 1$, $\dot{f}(t) \leqslant 0$ $(t_0 < t < t_*)$, and $f(t_*) = 0]$.*

In both problems, we assume that, for $t < 0$, the body was in a nondeformed state, and, hence, according to (1.1) and (1.4), for viscous strains ε^c_{kl}, everywhere in V we have

$$\varepsilon^c_{kl}\Big|_{t=0} = 0. \tag{1.8}$$

The above problems are considered within the framework of a conventional quasi-static formulation, which is generally adopted in the creep theory for metallic materials [2–4] when dynamic terms are not incorporated in equilibrium equations. In this case, the mass forces are assumed to be absent and, hence, the aforementioned equations have the form [4]

$$\sigma_{kl,l} = 0. \tag{1.9}$$

Thus, the system of equations for both problems includes the constitutive equations (1.1) and (1.4), the Cauchy equations (1.3), and the equilibrium equation (1.9). The initial conditions are of the form of (1.8), and the desired boundary conditions are present in the formulations of Problems 1 and 2. Note that for $t_0 = t_*$, the latter coincide with problems 1 and 2 from [4] and correspond to the case of instantaneous elastic unloading at moment $t = t_*$.

2. Auxiliary Assumptions. We denote the current residual displacements by $\tilde{u}_k$ which would remain in the region V at moment t considered after instantaneous removal of the current external loads p_k on S and also after elastic unloading. According to (1.3), to these displacements correspond the residual strains $\tilde{\varepsilon}_{kl}$; moreover,

$$\tilde{\varepsilon}_{kl} = a_{klmn}\rho_{mn} + \varepsilon^c_{kl}, \tag{2.1}$$

where ρ_{kl} are the residual stresses which appear in the body at moment t after unloading. Here [3, 4]

$$\sigma_{kl} = \sigma^e_{kl} + \rho_{kl}. \tag{2.2}$$

In the above relation, σ^e_{kl} are the components of the stresses corresponding to the solution of a purely elastic problem with the same loads p_k on S at the same moment t, so that to the stresses ρ_{kl} correspond to zero loads on S.

We shall introduce the notation for the quantities which are often used in our consideration:

$$I_1(\sigma_{kl}) = \Big(\int\limits_V \frac{1}{2} a_{klmn}\sigma_{kl}\sigma_{mn}dV\Big)^{1/2},\ I_2(\varepsilon_{kl}) = \Big(\int\limits_V \frac{1}{2} b_{klmn}\varepsilon_{kl}\varepsilon_{mn}dV\Big)^{1/2},\ I_3(t) = \int\limits_V \Delta\eta_{kl}\Delta\sigma_{kl}dV,$$

$$I_4(t) = \Big(\int\limits_0^t I_2^2(\Delta\eta_{kl})dt\Big)^{1/2},\ I_5(t) = \Big(\int\limits_0^t I_1^2(\Delta\sigma^e_{kl})dt\Big)^{1/2} = \Big(\int\limits_0^t \|\Delta \mathbf{u}^e\|^2 dt\Big)^{1/2},\ g(t) = \int\limits_0^t f^2 dt,$$

$$g_1(t) = \int\limits_0^t f\,dt, \qquad \beta(t) = \exp(-\lambda t)\int\limits_0^t |\dot{\varphi}|\exp(\lambda t)\,dt, \qquad \beta_0 = \beta(t_0),$$

$$\gamma = \lambda^{-1}\lambda_1^{-1}, \quad c_1 = \lambda[g_1(t_0) - (t_* - t_0)^{1/2}[\gamma^2 g(t_*) - g(t_0)]^{1/2}], \quad c_2 = \lambda_1^{-1}\sqrt{t_* g(t_*)},$$

$$c_3 = 1 - \beta_0 - [(t_* - t_0)(c_4 - \lambda^2 c_5)]^{1/2}, \quad c_4 = \lambda_1^{-2}(c_5 + c_6), \quad c_5 = \int\limits_0^{t_0} \beta^2\,dt, \quad c_6 = \beta_0^2\int\limits_{t_0}^{t_*} f^2\,dt,$$

$$c_7 = \int\limits_0^{t_0}\beta\,dt, \quad c_8 = \frac{1-\beta_0^2}{2\lambda}, \quad c_9 = \gamma^{-2}\beta_0^{-2}\Big[\frac{(1-\beta_0)^2}{\lambda^2(t_* - t_0)} - (\gamma^2 - 1)c_8\Big],$$

$$c_{10} = \lambda(\gamma - 1)(c_8 + c_6)\{1 - \beta_0 - \lambda(t_* - t_0)^{1/2}[(\gamma^2 - 1)c_8 + \gamma^2 c_6]^{1/2}\}^{-2}.$$

Let $\bar{\sigma}^e_{kl}$ be the stress field and $\bar{\varepsilon}^e_{kl} = a_{klmn}\bar{\sigma}^e_{kl}$ be the field of strains corresponding to the solution of the elastic problem under displacements u_k given on S. As is known [1], the quantity $\|\mathbf{u}\| = I_1(\bar{\sigma}^e_{kl}) = I_2(\bar{\varepsilon}^e_{kl})$ can serve as a norm for the displacement field if one excludes from the latter the displacement of a body as a rigid whole. In other words, the quantity $\|\mathbf{u}\|$ is the norm for the field $\mathbf{w} = \mathbf{u} - \mathbf{\Pi u}$, where $\mathbf{\Pi}$ is the operator of the orthogonal projection [relative to the scalar product in $(L^2(V))^3$] onto a set of rigid displacements. In what follows, by $\mathbf{u}$ we mean a set of displacements with subtraction of body displacement as a rigid whole, i.e., the field $\mathbf{w}$. If the point is, therefore, the uniqueness of solution of the corresponding problem in terms of displacements (Problem 2), by this we assume the uniqueness of determination of the field $\mathbf{w}$. The norm indicated above is produced by the following scalar product:

$$(\mathbf{u}_1, \mathbf{u}_2) = \int\limits_V \frac{1}{2} a_{klmn} \bar{\sigma}_{kl}^{e(1)} \bar{\sigma}_{mn}^{e(2)} \, dV = \int\limits_V \frac{1}{2} b_{klmn} \bar{\varepsilon}_{kl}^{e(1)} \bar{\varepsilon}_{mn}^{e(2)} \, dV.$$

Note that to the stresses σ_{kl}^e from (2.2) correspond the elastic displacements u_k^e (solution of the elastic problem under external loads p_k specified on S), so that $\|\mathbf{u}^e\| = I_1(\sigma_{kl}^e) = I_2(\varepsilon_{kl}^e)$, where $\varepsilon_{kl}^e = a_{klmn}\sigma_{mn}^e$.

Let the components of the displacement vector $u_k \in H^{1/2}(S)$ be given on S (the spaces used hereinafter are defined in [1]). A solution of the corresponding elastic problem then exists in the region V. Note that $u_k \in H^1(V)$. In this case, we obtain the following distribution of external loads on S: $p_k \in H^{-1/2}(S)$. [And vice versa, if $p_k \in H^{-1/2}(S)$ are specified on S, there exists the solution $u_k \in H^1(V)$ of the elastic problem.] We can set the norm $\|\mathbf{u}\| = I_1(\bar{\sigma}_{kl}^e) = I_2(\bar{\varepsilon}_{kl}^e)$ into correspondence to each displacement vector $\mathbf{u}$ on S. As is known from [1], under the adopted assumption that rigid displacement are absent the norm $\|\mathbf{u}\|_{H^1(V)}$ is equivalent to $\|\mathbf{u}\|$. Duvaut and Lions [1] established the inequality $\|\mathbf{u}\| \geqslant A\|\mathbf{w}\|_{H^1(V)}$, where $A > 0$;

$$\|\mathbf{w}\|_{H^1(V)}^2 = \int\limits_V (w_k w_k + w_{k,l} w_{k,l}) \, dV.$$

Owing to the positive definiteness of the matrix corresponding to the elastic-modulus tensor b_{klmn}, one can readily show that the inverse inequality $\|\mathbf{u}\| \leqslant A_1\|\mathbf{w}\|_{H^1(V)}$, $A_1 > 0$ also holds. In this connection, the quantities $u_k \in H^{1/2}(S)$ can be treated as the elements of the Hilbert space $\mathbf{U}$ with the norm $\|\mathbf{u}\|$. Similarly, the norm for the vector of external loads $\mathbf{p}$ on S is introduced: $\|\mathbf{p}\| = I_1(\sigma_{kl}^e) = I_2(\varepsilon_{kl}^e) = \|\mathbf{u}^e\|$, i.e., the quantities $p_k \in H^{-1/2}(S)$ can be regarded as the elements of the Hilbert space $\mathbf{P}$ with the above norm. Note that similar norms were used by Kuz'menko [5].

In our further consideration, we shall employ the equation of virtual work which, in the absence of mass forces, has the form [3]

$$\int\limits_V \sigma_{kl}\varepsilon_{kl} \, dV = \int\limits_S p_k u_k \, dS, \tag{2.3}$$

where the field σ_{kl} satisfies the equilibrium equations (1.9), $p_k = \sigma_{kl}n_l$ (n_k are the components of the unit vector of the normal which is external to S), and the fields ε_{kl} and u_k satisfy relations (1.3); here the quantities σ_{kl} and ε_{kl} are not related to each other.

By virtue of (2.3), we have

$$(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{2}\int\limits_S u_k^{(1)} \bar{p}_k^{(2)} \, dS = \frac{1}{2}\int\limits_S u_k^{(2)} \bar{p}_k^{(1)} \, dS.$$

Here $\bar{p}_k^{(i)} \in H^{-1/2}(S)$ are the external loads corresponding to the solution of the elastic problem in terms of displacements $u_k^{(i)} \in H^{1/2}(S)$, i.e., $\bar{p}_k^{(i)} = \bar{\sigma}_{kl}^{e(i)} n_l$ on S ($i = 1$ and 2). On the other hand, for arbitrary $p_k^{(1)} \in H^{-1/2}(S)$ and $u_k^{(2)} \in H^{1/2}(S)$, we have

$$\frac{1}{2}\int\limits_S u_k^{(2)} p_k^{(1)} \, dS = (\mathbf{u}_2, \mathbf{u}_1^e),$$

where $\mathbf{u}_1^e$ is the vector of elastic displacements over S, which corresponds to the vector of external loads $\mathbf{p}_1$.

To denote the increments of the corresponding quantities, we use the symbol Δ, as in Sec. 1. Note that, from (2.2) and (2.3), it follows that

$$I_1^2(\Delta\sigma_{kl}) = I_1^2(\Delta\sigma_{kl}^e) + I_1^2(\Delta\rho_{kl}) = \|\Delta\mathbf{u}^e\|^2 + I_1^2(\Delta\rho_{kl}) \geqslant \|\Delta\mathbf{u}^e\|^2, \tag{2.4}$$

since [3]

$$\int\limits_V a_{klmn}\Delta\sigma_{mn}^e \Delta\rho_{kl} \, dV = 0. \tag{2.5}$$

By virtue of (1.5), (1.6), and (2.4), we write

$$I_3 \geqslant 2\lambda I_1^2(\Delta\sigma_{kl}) \geqslant 2\lambda\|\Delta\mathbf{u}^e\|^2; \tag{2.6}$$

$$I_3 \geqslant 2\lambda_1 I_2^2(\Delta\eta_{kl}). \tag{2.7}$$

We then obtain estimates for $I_4(t)$ through $\|\Delta\mathbf{u}^e\|$. From (2.1) and (2.3), it follows that

$$\int\limits_S \Delta\dot{u}_k\Delta p_k\,dS = \int\limits_V \Delta\dot{\varepsilon}_{kl}\Delta\sigma_{kl}\,dV = \int\limits_V \Delta\dot{\varepsilon}_{kl}\Delta\sigma_{kl}^e\,dV.$$

With allowance for (2.1), (2.2), and (2.5), the last equality takes the form

$$\int\limits_V a_{klmn}\Delta\dot{\rho}_{mn}\Delta\rho_{kl}\,dV + I_3 = \int\limits_V \Delta\eta_{kl}\Delta\sigma_{kl}^e\,dV.$$

Integrating over time from zero to the instant t and taking into account that $\Delta\rho_{kl} = 0$ for $t = 0$ everywhere in V, from the above inequality we find that

$$\int\limits_0^t I_3\;dt = \int\limits_0^t\int\limits_V \Delta\eta_{kl}\Delta\sigma_{kl}^e\,dV dt - I_1^2\big(\Delta\rho_{kl}(t)\big) \leqslant \int\limits_0^t\int\limits_V a_{klmn}\Delta\sigma_{mn}^e(b_{klij}\Delta\eta_{ij})\,dV dt$$

$$\leqslant 2\Big(\int\limits_0^t I_1^2(\Delta\sigma_{kl}^e)\,dt\Big)^{1/2}\Big(\int\limits_0^t I_1^2(b_{klmn}\Delta\eta_{mn})\,dt\Big)^{1/2} = 2I_5(t)I_4(t), \tag{2.8}$$

where equalities (1.7) and the known inequality

$$\int\limits_0^t\int\limits_V a_{klmn}x_{kl}y_{mn}\;dV dt \leqslant \Big(\int\limits_0^t I_1^2(x_{kl})\,dt\Big)^{1/2}\Big(\int\limits_0^t I_1^2(y_{kl})\,dt\Big)^{1/2}$$

were used for $x_{kl} = \Delta\sigma_{kl}^e$ and $y_{kl} = b_{klmn}\Delta\eta_{mn}$.

On the other hand, from (2.6) and (2.7) we have

$$\int\limits_0^t I_3\,dt \geqslant 2\lambda I_5^2(t), \qquad \int\limits_0^t I_3\,dt \geqslant 2\lambda_1 I_4^2(t),$$

from which, with allowance for (2.8), we obtain

$$\lambda I_5(t) \leqslant I_4(t) \leqslant \lambda_1^{-1} I_5(t). \tag{2.9}$$

In addition, in our further considerations we shall need to estimate the increments of the residual displacements over S, i.e., the quantity $\|\Delta\tilde{\mathbf{u}}\|$ through $\|\Delta\mathbf{u}^e\|$. We shall use $\Delta\tilde{\sigma}_{kl}^e$ and $\Delta\tilde{p}_k = \Delta\tilde{\sigma}_{kl}^e n_l$ to denote the elastic stresses and the related external loads which correspond to the displacements $\Delta\tilde{u}_k$ over S. Similarly to (2.8), with allowance for (2.1) and (2.3) we then write

$$\frac{d}{dt}(\|\Delta\tilde{\mathbf{u}}\|^2) = \int\limits_V a_{klmn}\Delta\dot{\tilde{\sigma}}_{mn}^e\Delta\tilde{\sigma}_{kl}^e\,dV = \int\limits_S \Delta\dot{\tilde{u}}_k\Delta\tilde{p}_k\,dS$$

$$= \int\limits_V (a_{klmn}\Delta\dot{\rho}_{mn} + \Delta\eta_{kl})\Delta\tilde{\sigma}_{kl}^e\,dV = \int\limits_V \Delta\eta_{kl}\Delta\tilde{\sigma}_{kl}^e\,dV \leqslant 2I_2(\Delta\eta_{kl})\|\Delta\tilde{\mathbf{u}}\|,$$

from which follows the inequality $\|\Delta\tilde{\mathbf{u}}\|^{\cdot} \leqslant I_2(\Delta\eta_{kl})$. Integrating this inequality over time from zero to t and taking into account that $\|\Delta\tilde{\mathbf{u}}\| = 0$ for $t = 0$, we find

$$\|\Delta\tilde{\mathbf{u}}\| \leqslant \int\limits_0^t I_2(\Delta\eta_{kl})\,dt \leqslant \sqrt{t}I_4(t) \leqslant \lambda_1^{-1}\sqrt{t}I_5(t), \tag{2.10}$$

where the Cauchy–Bunyakovskii inequality and (2.9) were used.

3. Correctness of Inverse Problems. The study of the correctness of Problems 1 and 2 will be based on inequalities (2.6), (2.7), (2.9), and (2.10) and also on the lower estimate for the quantities $(\Delta\tilde{\mathbf{u}}_*, \Delta\mathbf{u}_0^e)$ and $(\Delta\tilde{\mathbf{u}}_*, \Delta\mathbf{u}_0)$, respectively (the subscripts asterisk and zero refer to the moments of time $t = t_*$ and $t = t_0$).

Lemma 1. *Let there be two sets of loads $p_k^{(i)} = f(t)p_{k0}^{(i)}$ ($i = 1$ and 2) which act on S, where the function $f(t)$ is subject to the conditions indicated in the statement of Problem 1. The inequality*

$$(\Delta\tilde{\mathbf{u}}_*, \Delta\mathbf{u}_0^e) \geqslant c_1\|\Delta\mathbf{u}_0^e\|^2 \tag{3.1}$$

then holds for the differences of the corresponding quantities.

Proof. Owing to (2.1)–(2.3), with allowance for the equalities $\Delta\tilde{u}_k\big|_{t=0} = 0$ we have

$$2(\Delta\tilde{\mathbf{u}}_*, \Delta\mathbf{u}_0^e) = \int_0^{t_*}\!\!\int_S \Delta\dot{\tilde{u}}_k \Delta p_{k0}\, dSdt = \int_0^{t_0}\!\!\int_S \frac{\Delta\dot{\tilde{u}}_k \Delta p_k}{f}\, dSdt + \int_{t_0}^{t_*}\!\!\int_S \Delta\dot{\tilde{u}}_k \Delta p_{k0}\, dSdt = I_6 + I_7,$$
$$I_6 = \int_0^{t_0} \frac{[I_1^2(\Delta\rho_{kl})]^{\cdot} + I_3(t)}{f}\, dt, \quad I_7 = \int_{t_0}^{t_*}\!\!\int_V (a_{klmn}\Delta\dot{\rho}_{mn} + \Delta\eta_{kl})\Delta\sigma_{kl0}^e\, dV dt = \int_{t_0}^{t_*}\!\!\int_V \Delta\eta_{kl}\Delta\sigma_{kl0}^e\, dV dt. \tag{3.2}$$

Let us estimate the quantities I_6 and I_7 from (3.2). For I_6, after integration by parts of the first term under the integral with allowance for the equalities

$$f(t_0) = 1, \qquad \lim_{t\to 0} \frac{I_1^2(\Delta\rho_{kl}(t))}{f(t)} = \lim_{t\to 0} \frac{[I_1^2(\Delta\rho_{kl}(t))]^{\cdot}}{\dot{f}(t)} = 0$$

[since $\dot{f}(0) > 0$ and $\Delta\rho_{kl}\big|_{t=0} = 0$], we obtain

$$I_6 = I_1^2(\Delta\rho_{kl0}) + \int_0^{t_0} [(\dot{f}/f^2) I_1^2(\Delta\rho_{kl}(t)) + I_3(t)/f]\, dt \geqslant 2\lambda g_1(t_0)\|\Delta\mathbf{u}_0^e\|^2. \tag{3.3}$$

Here the inequality holds true, because $\dot{f} \geqslant 0$ $(0 < t \leqslant t_0)$ and also by virtue of (2.6), where $\|\Delta\mathbf{u}^e\| = f\|\Delta\mathbf{u}_0^e\|$.

Similarly to (2.8), for I_7, recalling (2.9) we write

$$|I_7| \leqslant 2\Big(\int_{t_0}^{t_*} I_2^2(\Delta\eta_{kl})\, dt\Big)^{1/2} \Big(\int_{t_0}^{t_*} \|\Delta\mathbf{u}_0^e\|^2\, dt\Big)^{1/2}$$
$$= 2(t_* - t_0)^{1/2}\|\Delta\mathbf{u}_0^e\|\,[I_4^2(t_*) - I_4^2(t_0)]^{1/2} \leqslant 2(t_* - t_0)^{1/2}\|\Delta\mathbf{u}_0^e\|^2\,[\lambda_1^{-2}g(t_*) - \lambda^2 g(t_0)]^{1/2}. \tag{3.4}$$

From (3.2)–(3.4), it follows (3.1).

Theorem 1. *Let $\tilde{\mathbf{u}}_* \in \mathbf{U}$ and the function $f = f(t)$ be such that the constant c_1 from (3.1) is positive. Then there exists a unique solution $\mathbf{p}_0 \in \mathbf{P}$ of Problem 1, the operator $\mathbf{p}_0 = \mathbf{p}_0(\tilde{\mathbf{u}}_*)$ being continuous.*

Proof. By virtue of Lemma 1, we have $c_1\|\Delta\mathbf{u}_0^e\|^2 \leqslant (\Delta\tilde{\mathbf{u}}_*, \Delta\mathbf{u}_0^e) \leqslant \|\Delta\tilde{\mathbf{u}}_*\|\,\|\Delta\mathbf{u}_0^e\|$ from which follows $\|\Delta\mathbf{u}_0^e\| \leqslant c_1^{-1}\|\Delta\tilde{\mathbf{u}}_*\|$, thus ensuring the uniqueness and continuity of the solution.

The proof of its existence is similar to that by the author [4] for Problem 1 and is as follows. Let us consider the sequence $\{\mathbf{u}_{e0}^n\}$ where

$$\mathbf{u}_{e0}^{n+1} = \mathbf{u}_{e0}^n - \varepsilon(\tilde{\mathbf{u}}_*^n - \tilde{\mathbf{u}}_*) \text{ on } S\ (n = 0, 1, 2, \ldots),\ \varepsilon = \text{const}; \tag{3.5}$$

$\mathbf{u}_{e0}^0$ is an arbitrary element from $\mathbf{U}$. From (2.10), we obtain that $\|\tilde{\mathbf{u}}_*^0\| \leqslant c_2\|\mathbf{u}_{e0}^0\|$, i.e., $\tilde{\mathbf{u}}_*^0 \in \mathbf{U}$. Clearly, for any n, the elements of the sequence (3.5) belongs to the space $\mathbf{U}$.

From (3.5), we find that $\mathbf{u}_{e0}^{m+1} - \mathbf{u}_{e0}^{n+1} = \mathbf{u}_{e0}^m - \mathbf{u}_{e0}^n - \varepsilon(\tilde{\mathbf{u}}_*^m - \tilde{\mathbf{u}}_*^n)$ from which, with allowance for (2.10) and (3.1), we have

$$\|\mathbf{u}_{e0}^{m+1} - \mathbf{u}_{e0}^{n+1}\|^2 = \|\mathbf{u}_{e0}^m - \mathbf{u}_{e0}^n\|^2 - 2\varepsilon(\mathbf{u}_{e0}^m - \mathbf{u}_{e0}^n, \tilde{\mathbf{u}}_*^m - \tilde{\mathbf{u}}_*^n) + \varepsilon^2\|\tilde{\mathbf{u}}_*^m - \tilde{\mathbf{u}}_*^n\|^2 \leqslant \delta_1^2\|\mathbf{u}_{e0}^m - \mathbf{u}_{e0}^n\|^2,$$
$$\delta_1^2 = 1 - 2\varepsilon c_1 + \varepsilon^2 c_2^2.$$

It is evident that for $\delta_1 < 1$, i.e., for $0 < \varepsilon < 2c_1c_2^{-2}$, the sequence (3.5) is fundamental. The maximum convergence rate corresponds to the value $\varepsilon = c_1c_2^{-2}$ when $\delta_1^2 = 1 - c_1^2c_2^{-2}$. Since the space $\mathbf{U}$ is full, there exists $\lim_{n\to\infty} \mathbf{u}_{e0}^n = \mathbf{u}_{e0} \in \mathbf{U}$. Here, according to (3.5), $\lim_{n\to\infty} \tilde{\mathbf{u}}_*^n = \tilde{\mathbf{u}}_*$. As noted in Sec. 2, the element $\mathbf{u}_{e0} \in \mathbf{U}$ determines uniquely the element $\mathbf{p}_0 \in \mathbf{P}$.

The iterative process (3.5) can serve as a basis for construction of approximate solutions of the problem in question. Since $\|\mathbf{u}_{e0}^{n+1} - \mathbf{u}_{e0}^n\| \leqslant \delta_1\|\mathbf{u}_{e0}^n - \mathbf{u}_{e0}^{n-1}\|$ $(\delta_1 < 1)$, it is easy to show [4] that the convergence rate of successive approximations to an exact solution is determined by the inequality $\|\mathbf{u}_{e0}^n - \mathbf{u}_{e0}\| \leqslant \delta_1^n(1 - \delta_1)^{-1}\|\mathbf{u}_{e0}^1 - \mathbf{u}_{e0}^0\|$.

The above proof that Problem 1 is correct is essentially based on the condition $c_1 > 0$ which imposes definite restrictions on the function $f = f(t)$. To clarify these restrictions, with allowance for relation (3.1) one can write the inequality $c_1 > 0$ as follows:

$$g_1^2(t_0) > (t_* - t_0)[\gamma^2 g(t_*) - g(t_0)], \qquad \gamma = \lambda^{-1}\lambda_1^{-1} \geqslant 1 \tag{3.6}$$

[the inequality $\gamma \geqslant 1$ follows from (2.9); evidently, the case $\gamma = 1$ corresponds to the linear dependences (1.4) when $\eta_{kl} = \lambda a_{klmn}\sigma_{mn}$ and, hence, $\sigma_{kl} = \lambda_1 b_{klmn}\eta_{mn}$, where $\lambda_1 = \lambda^{-1}$; here the equality sign holds in (1.5) and (1.6)].

Since, owing to the Cauchy–Bunyakovskii inequality, $g_1^2(t_0) \leqslant t_0 g(t_0)$, we have the following necessary condition for satisfaction of (3.6): $g_0/g_* > \gamma^2(1 - t_0/t_*)$. This is possible only for

$$t_0/t_* > 1 - \gamma^{-2}, \tag{3.7}$$

so that $g_0 \leqslant g_*$. Inequality (3.7) establishes the lower bound for the moment when unloading begins.

We shall give an example of the function $f = f(t)$ for which the basic condition (3.6) is satisfied. Let

$$f(t) = \begin{cases} (t/t_0)^{\alpha_1} & (0 \leqslant t \leqslant t_0,\ 0 < \alpha_1 \leqslant 1), \\ [(t_* - t)/(t_* - t_0)]^{\alpha_2} & (t_0 < t \leqslant t_*,\ \alpha_2 > 0). \end{cases}$$

Condition (3.6) is then equivalent to the inequality $t_0/t_* > (1+\xi)^{-1}$, where

$$\xi = æ\left(\sqrt{1 + æ^{-2}(\alpha_1+1)^{-2}} - 1\right) \quad \left[æ = \frac{(\gamma^2 - 1)(2\alpha_2 + 1)}{[2\gamma^2(2\alpha_1 + 1)]}\right].$$

Naturally, this increases the lower bound for t_0 as compared with (3.7), because $(1+\xi)^{-1} > 1 - \gamma^{-2}$.

We shall consider Problem 2.

Lemma 2. *Let two sets of external actions be given on S:*

$$u_k^{(i)} = \varphi(t)u_{k0}^{(i)} \quad (0 \leqslant t \leqslant t_0), \quad p_k^{(i)} = f(t)p_{k0}^{(i)} \quad (t_0 \leqslant t \leqslant t_*) \quad (i = 1, 2),$$

where the functions $\varphi(t)$ and $f(t)$ are subject to the conditions formulated in Problem 2. The estimate

$$(\Delta\tilde{\mathbf{u}}_*, \Delta\mathbf{u}_0) \geqslant c_3\|\Delta\mathbf{u}_0\|^2 \tag{3.8}$$

is then true.

Proof. Since $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{u}^e$,

$$(\Delta\tilde{\mathbf{u}}_*, \Delta\mathbf{u}_0) = (\Delta\mathbf{u}_0 - \Delta\mathbf{u}_0^e, \Delta\mathbf{u}_0) + \int_{t_0}^{t^*} (\Delta\dot{\tilde{\mathbf{u}}}, \Delta\mathbf{u}_0)\,dt. \tag{3.9}$$

Let us estimate both terms on the right-hand side of equality (3.9). At any moment t $(0 \leqslant t \leqslant t_0)$, owing to (1.1) and (1.7) and also to the equality

$$\frac{1}{2}\int_V a_{klmn}\Delta\dot{\sigma}_{mn}\Delta\sigma_{kl}\,dV = \dot{I}_1(\Delta\sigma_{kl})I_1(\Delta\sigma_{kl})$$

we have

$$\dot{I}_1(\Delta\sigma_{kl})I_1(\Delta\sigma_{kl}) + \frac{1}{2}I_3(t) = \frac{1}{2}\int\limits_V \Delta\dot{\varepsilon}_{kl}\Delta\sigma_{kl}\,dV$$

$$= \frac{1}{2}\int\limits_V a_{klmn}\Delta\sigma_{mn}(b_{klij}\Delta\dot{\varepsilon}_{ij})\,dV \leqslant I_1(\Delta\sigma_{kl})I_2(\Delta\dot{\varepsilon}_{kl}).$$

Hence, taking into account (2.6) and the equality $I_2(\Delta\dot{\varepsilon}_{kl}) = \|\Delta\dot{\mathbf{u}}\| = |\dot{\varphi}|\,\|\Delta\mathbf{u}_0\|$, we obtain $\dot{I}_1(\Delta\sigma_{kl}) + \lambda I_1(\Delta\sigma_{kl}) \leqslant |\dot{\varphi}|\,\|\Delta\mathbf{u}_0\|$ or $d/dt[I_1(\Delta\sigma_{kl})\exp(\lambda t)] \leqslant \|\Delta\mathbf{u}_0\|\,|\dot{\varphi}|\exp(\lambda t)$.

Integrating this inequality over time from zero to the instant t and taking into consideration that $\Delta\sigma_{kl}\big|_{t=0} = 0$, we find that $I_1(\Delta\sigma_{kl}) \leqslant \beta(t)\,\|\Delta\mathbf{u}_0\|$. From (2.4), it then follows that

$$\|\Delta\mathbf{u}^e(t)\| \leqslant \beta(t)\,\|\Delta\mathbf{u}_0\| \qquad (0 \leqslant t \leqslant t_0). \tag{3.10}$$

In particular, $\|\Delta\mathbf{u}_0^e\| \leqslant \beta_0\,\|\Delta\mathbf{u}_0\|$, therefore $(\Delta\mathbf{u}_0 - \Delta\mathbf{u}_0^e, \Delta\mathbf{u}_0) \geqslant \|\Delta\mathbf{u}_0\|^2 - \|\Delta\mathbf{u}_0^e\|\,\|\Delta\mathbf{u}_0\| \geqslant (1 - \beta_0)\|\Delta\mathbf{u}_0\|^2$.

Similarly to the quantity I_7 from (3.2), for the integral in (3.9), with allowance for (3.10) we write

$$\Big|\int\limits_{t_0}^{t_*}(\Delta\dot{\mathbf{u}}, \Delta\mathbf{u}_0)\,dt\Big| \leqslant \Big(\int\limits_{t_0}^{t_*} I_2^2(\Delta\eta_{kl})\,dt\Big)^{1/2}\Big(\int\limits_{t_0}^{t_*}\|\Delta\mathbf{u}_0\|^2\,dt\Big)^{1/2}$$

$$\leqslant \|\Delta\mathbf{u}_0\|(t_* - t_0)^{1/2}\big[\lambda_1^{-2}I_5^2(t_*) - \lambda^2 I_5^2(t_0)\big]^{1/2} \leqslant \|\Delta\mathbf{u}_0\|^2(t_* - t_0)^{1/2}(c_4 - \lambda^2 c_5)^{1/2},$$

since $\|\Delta\mathbf{u}^e(t)\| = f(t)\|\Delta\mathbf{u}_0^e\|$ $(t_0 \leqslant t \leqslant t_*)$. From these inequalities and (3.9) follows (3.8).

Theorem 2. *Let $\tilde{\mathbf{u}}_* \in \mathbf{U}$ and the functions $\varphi = \varphi(t)$ and $f = f(t)$ be such that the constant c_3 from (3.8) is positive. Then there exists a unique solution $\mathbf{u}_0 \in \mathbf{U}$ of Problem 2, the operator $\mathbf{u}_0 = \mathbf{u}_0(\tilde{\mathbf{u}}_*)$ being continuous.*

Proof. The uniqueness and continuity of the solution are ensured by an inequality which follows from Lemma 2: $\|\Delta\mathbf{u}_0\| \leqslant c_3^{-1}\|\Delta\tilde{\mathbf{u}}_*\|$. To prove its existence, note that, from (2.10) and (3.10), it is not difficult to obtain

$$\|\Delta\tilde{\mathbf{u}}_*\| \leqslant \sqrt{c_4 t_*}\,\|\Delta\mathbf{u}_0\|. \tag{3.11}$$

Let us consider the sequence $\{\mathbf{u}_0^n\}$, which is similar to (3.5):

$$\mathbf{u}_0^{n+1} = \mathbf{u}_0^n - \varepsilon(\tilde{\mathbf{u}}_*^n - \tilde{\mathbf{u}}_*) \text{ on } S \ (n = 0, 1, 2, \ldots), \quad \varepsilon = \text{const} \tag{3.12}$$

($\mathbf{u}_0^0$ is an arbitrary element from $\mathbf{U}$). With allowance for (3.8) and (3.11), we therefore have

$$\|\mathbf{u}_0^{m+1} - \mathbf{u}_0^{n+1}\| \leqslant \delta_2\|\mathbf{u}_0^m - \mathbf{u}_0^n\|, \quad \delta_2^2 = 1 - 2\varepsilon c_3 + \varepsilon^2 c_4 t_*.$$

Hence, for $0 < \varepsilon < 2c_3/(c_4 t_*)$, the sequence (3.12) is fundamental and, owing to the fullness of the space $\mathbf{U}$, converges to the element $\mathbf{u}_0 \in \mathbf{U}$; here $\lim_{n\to\infty}\tilde{\mathbf{u}}_*^n = \tilde{\mathbf{u}}_*$.

The iterative process (3.12) can be used to find approximate solutions of the problem considered; here $\|\mathbf{u}_0^n - \mathbf{u}_0\| \leqslant \delta_2^n(1 - \delta_2)^{-1}\|\mathbf{u}_0^1 - \mathbf{u}_0^0\|$ $(\delta_2 < 1)$.

Let us consider limitations which are imposed on the functions $\varphi = \varphi(t)$ and $f = f(t)$ by the condition $c_3 > 0$. From this condition, it follows that $\beta_0 < 1$, which is the case, for example, if $\dot{\varphi}(t) \geqslant 0$. Indeed, in this case

$$\beta_0 = \exp(-\lambda t_0)\int\limits_0^{t_0}\dot{\varphi}\exp(\lambda t)\,dt = 1 - \lambda\exp(-\lambda t_0)\int\limits_0^{t_0}\varphi\exp(\lambda t)\,dt < 1,$$

because $\varphi(0) = 0$, $\varphi(t_0) = 1$, and $0 < \varphi(t) < 1$ $(0 < t < t_0)$. Therefore, it follows, in particular, that the minimum value of β_0 corresponds to a relaxation regime of deformation in the interval $[0, t_0]$ when $\varphi(0) = 0$,

$\dot\varphi(t) > 0$ $(0 < t < t_1)$, $\varphi(t) = 1$ $(t_1 \leqslant t \leqslant t_0)$ for $t_1 \to 0$. It is easy to see that $\beta_{0\,\min} = \exp(-\lambda t_0)$; i.e., in this case

$$\int_0^{t_0} |\dot\varphi| \exp(\lambda t)\, dt = 1,$$

while, for any other regime,

$$\int_0^{t_0} |\dot\varphi| \exp(\lambda t)\, dt \geqslant \int_0^{t_0} |\dot\varphi|\, dt \geqslant \int_0^{t_0} \dot\varphi\, dt = 1.$$

With allowance for (3.8), the inequality $c_3 > 0$ can be represented as

$$\int_{t_0}^{t_*} f^2\, dt < \gamma^{-2}\beta_0^{-2}\Big[\frac{(1-\beta_0)^2}{\lambda^2(t_* - t_0)} - (\gamma^2 - 1)c_5\Big]. \tag{3.13}$$

Assuming hereinafter that $\dot\varphi \geqslant 0$, we obtain the lower and upper estimates for the constant c_5 from (3.13). Owing to the Cauchy–Bunyakovskii inequality, we have $c_5 \geqslant c_7^2 t_0^{-1}$ and

$$c_7 = \int_0^{t_0} \exp(-\lambda t)\Big(\int_0^t \dot\varphi \exp(\lambda t)\, dt\Big)\, dt = -\lambda^{-1}\Big[\exp(-\lambda t)\int_0^t \dot\varphi \exp(\lambda t)\, dt - \varphi(t)\Big]_0^{t_0} = \lambda^{-1}(1 - \beta_0).$$

Here the procedure of integration by parts was used. Using this procedure, we also can show that

$$c_5 = -\frac{(\beta - \varphi)^2}{2\lambda}\Big|_0^{t_0} + \int_0^{t_0} \varphi\beta\, dt \leqslant -\frac{(1-\beta_0)^2}{2\lambda} + c_7 = c_8.$$

Thus, we have

$$(1-\beta_0)^2/(\lambda^2 t_0) \leqslant c_5 \leqslant c_8. \tag{3.14}$$

From the first inequality in (3.14), we obtain a condition which is necessary to satisfy (3.13): $(t_* - t_0)^{-1} - (\gamma^2 - 1)t_0^{-1} > 0$. This condition is equivalent to inequality (3.7) and yields the same lower bound for the moment t_0 of onset of unloading as in Problem 1.

The sufficient condition for satisfaction of (3.13) follows from the second inequality in (3.14):

$$\int_{t_0}^{t_*} f^2\, dt < c_9. \tag{3.15}$$

Inequality (3.15) is possible if $c_9 > 0$, i.e., for $2(1-\beta_0)/[(\gamma^2 - 1)(1+\beta_0)] > \lambda(t_* - t_0)$.

With the function $\varphi = \varphi(t)$ specified in the active-loading interval, i.e., for $0 \leqslant t \leqslant t_0$, the conditions (3.13) and (3.15) can be regarded as restrictions on the function $f = f(t)$ under unloading $(t_0 \leqslant t \leqslant t_*)$. Hence, if $f(t) = [(t_* - t)/(t_* - t_0)]^\alpha$ $(t_0 \leqslant t \leqslant t_*)$, inequality (3.15) is satisfied for $\alpha > [c_9^{-1}(t_* - t_0) - 1]/2$.

4. Estimates of the Level of Residual Stresses for $t = t_*$. The inequalities which were used in proving Theorems 1 and 2 allow one to obtain, for each problem considered, the upper estimates for the level of residual stresses in the body at moment $t = t_*$ after unloading. As a measure that characterizes this level, we choose the quantity

$$I_8 = \frac{1}{2} I_1^2(\rho_{kl*}) + \lambda \int_0^{t_*} I_1^2(\rho_{kl}(t))\, dt.$$

It should be noted that the formulas that we derived above for the differences of the quantities that characterize two states are also valid for the quantities of the basic state, because this state can be chosen as the first state, while the natural state corresponding to zeroth displacements, strains, and stresses throughout the region V can be used as the second state.

We introduce the following notation:

$$I_9 = \int\limits_0^{t_*} \|\mathbf{u}^e\|^2\, dt, \qquad I_{10} = \Big(\int\limits_0^{t_*} I_2^2(\eta_{kl})\, dt\Big)^{1/2}.$$

Then, from (2.3), with allowance for (2.4), (2.6), and (2.9), similarly to (3.2) we obtain

$$I_8 + \lambda I_9 \leqslant \int\limits_0^{t_*} (\dot{\tilde{\mathbf{u}}}, \mathbf{u}^e)\, dt = \frac{1}{2}\int\limits_0^{t_*}\int\limits_V \eta_{kl}\sigma_{kl}^e\, dV dt \leqslant I_9^{1/2} I_{10} \leqslant \lambda_1^{-1} I_9,$$

from which

$$I_8 \leqslant (\lambda_1^{-1} - \lambda) I_9. \tag{4.1}$$

Since $\|\mathbf{u}^e\| = f\|\mathbf{u}_0^e\|$ and $\|\mathbf{u}_0^e\| \leqslant c_1^{-1}\|\tilde{\mathbf{u}}_*\|$ in Problem 1, from (4.1) it follows that

$$I_8 \leqslant (\lambda_1^{-1} - \lambda) c_1^{-2} g_* \|\tilde{\mathbf{u}}_*\|^2. \tag{4.2}$$

Because $f(t) \geqslant 0$ $(0 \leqslant t \leqslant t_*)$, one can see from (3.1) that $c_1 \leqslant \lambda g_1(t_*)$, the equality sign occurring only for $t_0 = t_*$. In view of this, from (4.2) it follows that the minimum estimate for I_8 is obtained in the case where the functional

$$\lambda^{-2} g(t_*)/g_1^2(t_*) = \lambda^{-2}\int\limits_0^{t_*} f^2\, dt \Big/ \Big(\int\limits_0^{t_*} f\, dt\Big)^2$$

reaches a minimum value in the set of functions $f = f(t)$ subject to the conditions formulated in Problem 1. Since $g_1^2(t_*) \leqslant t_* g(t_*)$, the equality sign being possible only for $f(t) = \text{const}$ on $[0, t_*]$, this minimum is equal to $\lambda^{-2} t_*^{-1}$ and corresponds to a stress at which the function $f = f(t)$ $[0, t_1]$ $(t_1 \to 0)$ increases monotonically (instantaneously) from 0 to 1, then $f(t) = 1$ for $t_1 < t < t_0$, and decreases monotonically from 1 to 0 in the interval $[t_0, t_*]$ $(t_0 \to t_*)$ (i.e., an instantaneous unloading occurs at $t = t_*$). In this case, from (4.2) we obtain $I_8 \leqslant (\gamma - 1)\lambda^{-1} t_*^{-1} \|\tilde{\mathbf{u}}_*\|^2$.

In Problem 2, we have $\|\mathbf{u}^e(t)\| \leqslant \beta(t)\|\mathbf{u}_0\|$ for $0 \leqslant t \leqslant t_0$ owing to (3.10) and $\|\mathbf{u}^e(t)\| = f(t)\|\mathbf{u}_0^e\|$ for $t_0 \leqslant t \leqslant t_*$ and $\|\mathbf{u}_0\| \leqslant c_3^{-1}\|\tilde{\mathbf{u}}_*\|$. Hence, from (4.1) we find $I_8 \leqslant (\lambda_1^{-1} - \lambda)\lambda_1^2 c_3^{-2} c_4 \|\tilde{\mathbf{u}}_*\|^2$ from which, with allowance for (3.8) and (3.14), we obtain a c_5-independent estimate:

$$I_8 \leqslant c_{10} \|\tilde{\mathbf{u}}_*\|^2. \tag{4.3}$$

If the moment $t = t_0$ at which unloading begins is fixed, as is seen from (4.3), to minimize the estimate obtained, it is necessary to set $f(t) = 0$ with $t_0 < t < t_*$, i.e., $c_6 = 0$. This corresponds to an instantaneous unloading at $t = t_0$ and to the period during which the surface S of the body in question is in an unloaded state in the interval (t_0, t_*). For $c_6 = 0$, the function $c_{10} = c_{10}(\beta_0)$ from (4.3) is an increasing one (for $c_3 > 0$, which was used in Sec. 3), therefore $\min c_{10} = c_{10}(\beta_{0\,\min})$. As mentioned above, $\beta_{0\,\min} = \exp(-\lambda t_0)$, which corresponds to a relaxation regime of deformation in the interval $[0, t_0]$.

If the magnitude of t_0 is not fixed, as it is easy to see, a minimum of the estimate (4.3) is reached for $t_0 = t_*$ and $\beta_0 = \beta_* \equiv \exp(-\lambda t_*)$, which corresponds to the aforementioned relaxation regime for $0 \leqslant t < t_*$ and to an instantaneous unloading for $t = t_*$. In this case, $\min c_{10} = (1/2)(\gamma - 1)(1 + \beta_*)(1 - \beta_*)^{-1}$.

5. Estimation of the Level of Residual Displacements for $t > t_*$. We assume that after unloading the body surface is in an unloaded state, i.e., $p_k = 0$ on S for $t > t_*$. However, since at moment $t = t_*$ the self-balanced (nonzero in the general case) residual stresses occurred in the region V, the residual strains $\tilde{\varepsilon}_{kl}$ and displacements $\tilde{u}_k$ will change $t > t_*$ owing to relaxation of these stresses. Let us estimate their level by choosing the quantity $\|\tilde{\mathbf{u}}(t)\|$ $(t > t_*)$ as a measure of the level.

The arguments which were used to derive (2.10) lead to the inequality

$$\|\tilde{\mathbf{u}}\|^{\cdot} \leqslant I_2(\eta_{kl}). \tag{5.1}$$

Bearing in mind that

$$\int_V \sigma_{kl}\eta_{kl}\,dV = \int_V a_{klmn}\sigma_{mn}(b_{klij}\eta_{ij})\,dV \leqslant 2I_1(\sigma_{kl})I_2(\eta_{kl}),$$

and taking into account (2.7) where the symbol Δ is omitted, we obtain $I_2(\eta_{kl}) \leqslant \lambda_1^{-1} I_1(\sigma_{kl})$. Together with (5.1), this gives

$$\|\tilde{\mathbf{u}}\|^{\cdot} \leqslant \lambda_1^{-1} I_1(\sigma_{kl}). \tag{5.2}$$

Since $\sigma^e_{kl} = 0$, i.e., $\sigma_{kl} = \rho_{kl}$ at $t > t_*$, from (2.1), (2.3), and (2.6) we find that

$$0 = \int_V (a_{klmn}\dot{\rho}_{mn}\rho_{kl} + \eta_{kl}\rho_{kl})\,dV \geqslant \frac{d}{dt}[I_1^2(\rho_{kl})] + 2\lambda I_1^2(\rho_{kl})$$

from which $[I_1(\rho_{kl})\exp(\lambda t)]^{\cdot} \leqslant 0$.

Integrating this inequality over time from t_* to the instant t, we obtain

$$I_1(\sigma_{kl}) = I_1(\rho_{kl}) \leqslant I_1(\rho_{kl*})\exp[\lambda(t_* - t)]. \tag{5.3}$$

Substituting (5.3) into (5.2) and integrating over t, we find that $\|\tilde{\mathbf{u}}(t)\| \leqslant \|\tilde{\mathbf{u}}_*\| + \gamma I_1(\rho_{kl*})(1-\exp[\lambda(t_* - t)]) \leqslant \|\tilde{\mathbf{u}}_*\| + \gamma I_1(\rho_{kl*})$ which yields the desired estimate.

The inequalities that we derived here and in Sec. 4 allow us to select a deformation regime in which the body will have the desired residual displacements on S at moment $t = t_*$ with a level of residual stresses and a further level of residual displacements for $t > t_*$ which do not exceed the given values.

In conclusion, we note that many of the results obtained can be extended to more general media of the form of (1.1) for which the velocities of viscous (creep strains) depend not only on stresses, as in (1.4), but also on a set of structural parameters whose rates of change are described by kinetic equations [2, 3]. In this case, the basic inequalities (1.5) and (1.6) should be replaced by

$$\int_0^t \Delta\eta_{kl}\Delta\sigma_{kl}\,dt \geqslant \lambda(t)\int_0^t a_{klmn}\Delta\sigma_{kl}\Delta\sigma_{mn}\,dt, \quad \lambda(t) > 0,$$

$$\int_0^t \Delta\eta_{kl}\Delta\sigma_{kl}\,dt \geqslant \lambda_1(t)\int_0^t b_{klmn}\Delta\eta_{kl}\Delta\eta_{mn}\,dt, \quad \lambda_1(t) > 0$$

subject to the corresponding initial conditions at moment $t = 0$ [4].

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